Street lighting,
what does it cost?

a mathematics project for secondary education

European Schoolnet

inGenious

Many parties were involved in making this project available for schools:

PHILIPS

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Jet-Net, the Dutch Youth and Technology Network, is a partnership between companies, education and government. The aim is to provide higher general secondary school (HAVO) and pre-university school (VWO) pupils with a true picture of science and technology and to interest them in a scientific-technological higher education course.

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1. Street lighting? What a bright idea!

1.1 Well lit!

Less than a hundred years ago it used to be pitch black on the streets at night. All you saw was the occasional gas lantern or flickering oil lamp, and the rest was in total darkness. Now that has all changed and endless rows of lampposts ensure that roads are illuminated fairly uniformly. When you drive along a road at night it can almost seem as if street lighting doesn't cost anything at all.

Nothing could be further from the truth, however. Lighting uses electricity, and electricity costs money. Every meter of illuminated road costs money each year. But how much does it cost? Ideally, as little as possible. When there are plans to build a new road, an engineer works out the cheapest way of lighting the road. The absolute cheapest way of all would, of course, be to have no street lamps whatsoever, but that is not a sensible option. A road needs to be safe, and that means it needs to be well lit after dark.

The question we are going to examine is this:
What is the minimum amount of money required to ensure a road is well lit?

1.2 A general problem approach

You have probably been faced with difficult questions before, perhaps even in your math class. The trouble with a difficult question is that it is often hard to know where to start. And that's where we're going to help you.
You will notice that there is a certain way to go about answering this question. At each stage we will explain the steps you need to take in a box like this one. These steps together make up the ***problem approach***. This problem approach is very general and that means you will also be able to use it again to answer another, totally different, question.
And that might just turn out to be very useful!

When you are faced with a difficult question to which you don't know the answer straight away, the first thing you need to do is to work out whether you understand the question. Is it clear what you are being asked to do? Do you fully understand all of the terms used? In our question, for example, you might want to make sure you understand what is meant by 'well lit'.
Sometimes it can help to make a number of ***assumptions***. We assume something, which may not necessarily have to *always* be true. When you make an assumption you make the problem more concrete and that often makes it a bit easier. Once you have solved the problem on the basis of a given assumption, you can then see if you can also solve it by making different assumptions. This may sound a bit complicated, but we will explain it in more detail in the text and tasks as we go along.

2. Lighting a road

2.1 Well lit?

A lot of lighting costs a lot of money. What's more, too much lighting is not a good thing. A road is well lit when there is a given *minimum light intensity* everywhere along it. The minimum light intensity is the minimum amount of light energy that is required to ensure that one square meter of road is well lit. From now on we will assume that a road is well lit if the light intensity at every point on the road is at least 10 watts per square meter. It will help if we also know how many euros one kWh of electrical energy costs and how many square meters the road surface measures.

Assumption 1: *A road is well lit if the light intensity everywhere along it is at least 10 watts per square meter.*

Tasks

1. Read the text on pages 1 and 2 of this booklet again.

   State two requirements that good street lighting should fulfill.

2. How can an engineer vary the intensity of the street lighting? State two things.

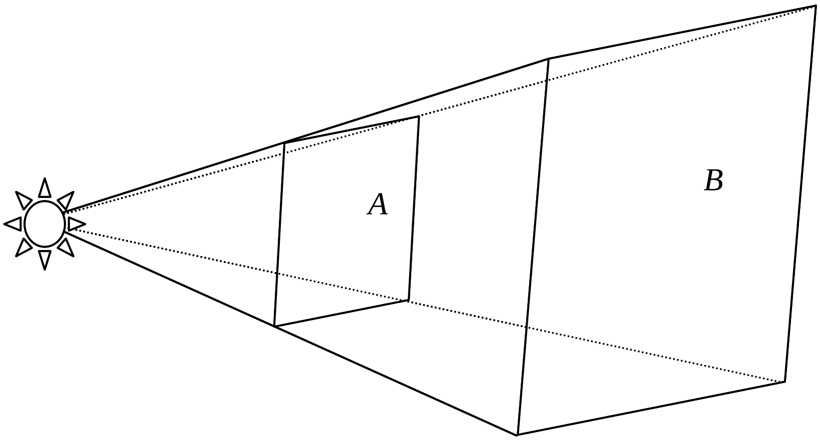
3. Why is too much lighting not a good thing? Give two reasons.

4. From now on we will assume that 1 kWh of electricity costs € 0.15.
   Calculate the minimum annual cost of illuminating one kilometer of road that is 8 meters wide. Assume that the lamps are switched on for, on average, 8 out of every 24 hours and that all of the light shines on the road.

Assumption 2: *One kWh of electricity costs € 0.15.*

2.2 Light spreads out

Unfortunately it is not that simple. You will no doubt have noticed that the area immediately below a street light is brighter than the area a short distance away from it. This is because the light spreads out over a large surface area. And the light is not always as uniform as we'd like it to be. The points right next to a light source receive a lot more watts per square meter than the points further away. That means there are light and dark patches on the road.



The diagram above shows a light source, e.g. a lamp. We are going to look at the light that the lamp shines through the square windows *A* and *B*. The light source is so strong that it shines a power of 10 watts through window *A*. This means that 10 joules of energy shine through window *A every second*. Window *B* is exactly twice as far away from the light source as window A, and the size of window *B* is such that all the light that shines through window *A* also shines through window *B*. This means that 10 watts shine through window *B* as well.

Tasks

5. Show that the surface area of window *B* is four times larger than the surface area of window *A*. Write down your explanation.
Tip: Think about magnification factors.

6. We compare the light intensity at window *A* and at window *B*.
How many times higher or lower is the light intensity at window *B* than at window *A*? Explain your answer.

7. Consider a light source that shines a power *P* through a surface area measuring 1 m^2 that is located 1 m away from this light source.
Show that at a distance *d* from this light source the light intensity *S* is given by the formula $S = \frac{P}{d^2}$.

Here *S* is the light intensity in watts/m^2, *P* is the power in watts and *d* is the distance in meters.

2.3 One street lamp

The formula we encountered in task 7 is a very important one. It tells us that the light intensity from a lamp depends on the distance. The further away a lamp is, the weaker the light intensity. That seems very logical, but it also means that however far away the lamp is there will always be a small amount of light coming from it. A little bit of the light from the street lamps here on earth even reaches the planet Mars. Even street lamps that are some distance away can contribute considerably to the light intensity close by. This makes the problem a bit more complicated, so let's first take a look at a situation in which there is only one street lamp.

The problem is a bit more complicated than we originally thought, so let's just go back a step, which will make the problem a bit less complicated. There are a number of ways of doing this. First of all, let's look at a situation in which there is only **one street lamp**. What do we find here?

We will also make the problem a bit more concrete. We will do that by making a number of assumptions. For example, in the text below we assume that the height of the street lamp is 5 meters and the power of the street lamp is 1000 watts.

Tasks

Imagine that you are standing directly below a street lamp that is 5 meters high. The street lamp has a power of 1000 watts. By that we mean that, at a distance of one meter, the street lamp has a power density of 1000 watts per m^2.

Assumption 3: *The height of a street lamp is 5 meters.*

Assumption 4: *The street lamp has a power of 1000 watts.*

8. Show that the light intensity on the road directly below the street lamp is equal to 40 watts/m^2

Now walk a few meters further away, parallel to the edge of the pavement. We call these few meters *x*.

9. Show that at the point where you are now standing the following applies: $d = x^2 + 25$
Now explain that the light intensity at this point is equal to:

$$S = \frac{1000}{x^2 + 25}$$

10. Complete the table shown below.

x	0	5	10	15	20
S					

Draw a graph with distance x and light intensity S.
Mark the distance x along the horizontal axis and the light intensity S along the vertical axis.

Let the x axis run from 0 to 75 meters and the y axis from 0 to 50 watts/m^2.

This is how to draw the iso light-intensity lines (these are lines or curves that join up points of equal light intensity).

2.4 There is always more than one street lamp

There is always more than one street lamp on a road. We are going to look at this now. Let us assume that the distance between two street lamps is always 30 meters.

Assumption 5: *The distance between two street lamps is always 30 meters.*

Now that we have considered the situation for one street lamp, we are going to make things a little bit more difficult. We shall add one street lamp each time. And we are going to make another assumption: the distance between the two street lamps is always 30 meters.

Tasks

11. In the graph for question 10, draw in the light intensity values for the street lamps that are 30 meters and 60 meters further away (i.e. at x=30 and x=60 in the graph).

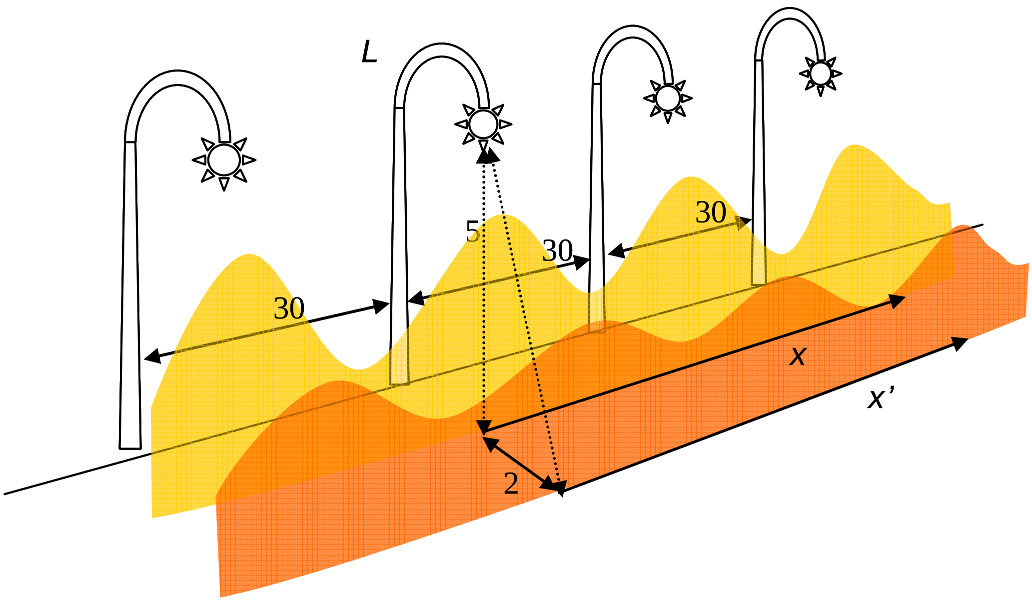
When you are standing on a road, light shines on you from all of the street lamps. The *total* light intensity at any given point is equal to the light intensity from every one of the individual street lamps added together.

12. In the graph for question 10, draw in the *total* light intensity at a given point. Use a different color for this curve to avoid any confusion.

2.5 In the middle of the road

Now that we are able to calculate the light intensity when you are walking directly under the street lamps, we can make the problem a bit more realistic, and that means more difficult. What if you are walking down the middle of the road? We are going to investigate this now. We shall base our calculations on a single street lamp. We don't want to make it too difficult to start with!

So far we have been walking directly below the street lamps. The light is then coming down onto us vertically from above. This is not always the case, of course. If you walk down the middle of the road and the street lamps are along the side of the road, the light from the street lamps will shine down on you at an angle.
The diagram below shows a row of street lamps. The distance between the street lamps is still 30 meters and the height of the street lamps is 5 meters.



You are now no longer standing directly below the street lamp but 2 meters out into the road, as shown in the diagram.

Tasks

13. Calculate the light intensity of street lamp *L* at the point where you are standing now.
Tip: Think about using Pythagoras

Now you walk in direction *x'*, parallel to the pavement.

14. What is the formula for calculating the light intensity for street lamp *L* at point *x'* ? Explain how you come up with this formula.
Tip: Use task 13 and Pythagoras again

15. What would be the difference between the graph you drew in question 10 and the graph to show the light intensity at point *x'* for street lamp *L*? You do not need to draw this graph.

2.6 There are two sides to a road

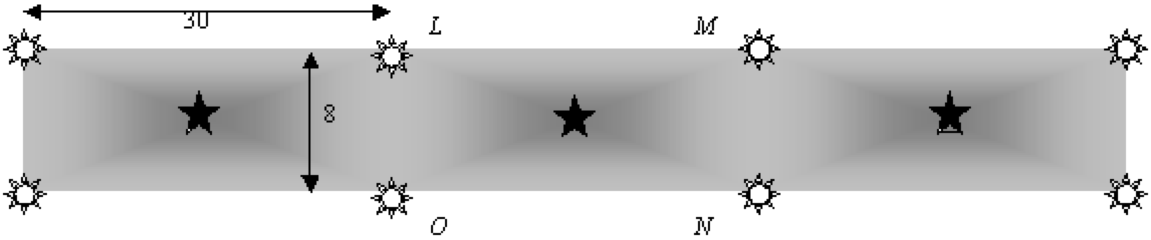
We already know quite a lot now. We have calculated what the total light intensity is if you take into account more than one street lamp. We have also calculated how the light intensity is affected if you walk out into the road so that you are no longer standing directly below a street lamp.
However, we have not finished yet. There are two sides to a road, and there are street lamps on the other side as well. Now we are going to take them into account too.
Let us assume that the street lamps are positioned directly opposite each other. We will assume the road is 8 meters wide.

Assumption 6: *The street lamps are positioned immediately opposite each other.*

Assumption 7: *The road is 8 meters wide.*

Now we are going to combine everything. We shall consider several street lamps at once and take into account the entire road, rather than just the edges.

As you have seen in the graph in question 12, the point between two street lamps is often the darkest. For the rest of this task we will assume that this is the case.
The figure below shows a view of the road from above. The shining circles are the street lamps. This is where the brightest points are. The darkest points are shown by a black star, in the middle of the road and at the midpoint between two street lamps.



Tasks

In the graph you drew for question 12 you will see that street lamp *M* (which is 30 meters away) barely contributes to the light intensity under street lamp *L*.

16. Explain how you can deduce from this that street lamp *N* also barely contributes to the light intensity directly below street lamp *L*.

17. Show that the contribution of street lamp *O* to the light intensity directly below street lamp *L* is equal to 11.2 watts/m^2.

18. Make a rough calculation of the maximum light intensity on this road.

19. Make a rough calculation of the minimum light intensity on this road as well.
Tip: First consider the contribution made by one street lamp. Calculate this contribution by applying Pythagoras twice.

The maximum and minimum light intensity on a road is governed by law. The legal minimum light intensity is 10 watts/m^2 and the legal maximum light intensity is 320 watts/m^2. We shall base our subsequent calculations on this.

Assumption 8: *The maximum light intensity is 320 watts/m^2.*

3. The cost

We have made quite a lot of progress, but we have barely mentioned the cost yet. What kind of costs are we dealing with here?

Electricity costs:
lighting uses electricity and electricity costs money. We saw earlier that electricity costs are calculated on the basis of kWh. One kWh costs € 0.15.

Street lamp costs:
lampposts and lamps have to be purchased. They also have to be replaced periodically because they break or become obsolete. These costs are referred to as *depreciation costs.* Maintenance also costs money. Street lamps need to be kept clean and they need to be painted regularly. We shall assume that each street lamp costs € 200 per year in maintenance and depreciation.

Material costs:
a lamppost is made of steel. There are also costs associated with the maintenance and depreciation of this. Let us assume that every meter of lamppost steel costs € 50 per year in maintenance and depreciation.

Assumption 9: *Every street lamp costs € 200 per year in maintenance and depreciation.*

Assumption 10: *Every meter of lamppost steel costs € 50 per year in maintenance and depreciation.*

Task

20. How much do the street lamps along a 1-km stretch of road cost in maintenance and depreciation?

3.1 Taking decisions

Now it is time we took some decisions. How far apart do the lampposts need to be in order to ensure it is still just light enough at the darkest points? And how close together can they be without making it too bright at the brightest points? How high do they need to be and what does the power of each street lamp need to be? We want to know all this in order to be able to answer our initial question:
What is the minimum amount of money required to ensure a road is well lit?

Engineers refer to problems like this one as *optimization problems*. These are problems where you need to choose the best possible (optimum) values for a number of quantities.
In our case these are:

- the height, *h* (in meters), of each street lamp;
- the power, *P* (in watts), of each lamp;
- the distance, *a* (in meters), between two adjacent street lamps.

In assumptions 3, 4 and 5 we have used fixed values for these quantities (5 meters for the height, 1000 watts for the power and 30 meters for the distance between two street lamps). We needed these fixed values in order to understand the problem. Now we are going to vary these values in order to see which values give rise to the lowest cost.

Now that we have looked at the problem from every angle and have a good understanding of it, we can get rid of a number of the assumptions we made. This will enable us to keep the problem as general as possible.

The values we choose will be the best possible values if two things hold:

1. The *basic conditions* must be fulfilled: the maximum light intensity must not exceed 320 watts/m^2, and the minimum light intensity must not be less than 10 watts/m^2;
2. The costs, per year and per (kilo)meter of road to be lit, must be as low as possible.

The value of *h*, *P* and *a* all affect the cost. There are three types of costs: electricity costs, street lamp costs and material costs.

Tasks

21. Which costs are affected by the height *h* of the street lamp? Explain how they are affected.

22. Which costs are affected by the power *P* of the lamp? Explain how they are affected.

23. Which costs are affected by distance *a* between two street lamps? Explain how they are affected.

If we look at everything together, the formula for the total cost in euros per year per meter of road is:
$C = (400 + 100h + 0.9P)/a$, where h and a are in meters and P is in watts

24. Calculate the cost per year per meter of road if $h = 5$ m, $a = 30$ m and $P = 1000$ W.

Engineers who design street lighting have to take many more factors into account. Not all lamps cost the same; if you buy a lot of electricity from the electricity company the price per kWh is lower; tall lampposts are more expensive to maintain than shorter lampposts; mirrors and lenses can be incorporated into the lamps in order to try and ensure the light is distributed more uniformly, and there are probably another 100 or so other factors too. You should therefore not be surprised that in real life the heights, distances and the power of street lamps may vary from the answers we find in this task. But … you will at least get some idea of how the decisions are made. And even our simplified approach will teach you a number of important things about street lighting.

You have also learned a lot about how to tackle a difficult problem. Read the text in the boxes through again.

25. Write a list of the things you could do if you are faced with a difficult problem.

4. The game

Now that we have a fair idea of how it all works, we are going to try it out. We are going to pretend that we have to design the street lighting for a section of freeway. To do this we will use a computer and a specially developed computer application: *street_lighting_what_does_it_cost*. In the box you will find instructions on how to use this computer application.

INSTRUCTIONS for using street_lighting_what_does_it_cost

Double click to start up the street_lighting_what_does_it_cost application. It is an EXCEL application, so you can also start it up in the MS-EXCEL™ program. If you have never used EXCEL before, ask your teacher for help.

The street_lighting_what_does_it_cost application enables you to enter your chosen values for *a*, *h*, and *P*. You can do this by dragging the three sliders at the top left of the work sheet. The light-green slider determines the distance *a*; the light-blue slider determines the height *h*, and the yellow slider determines the power *P*. You can also type in numbers manually instead of using the sliders.

Once you have selected the values for *a*, *h*, and *P*, click on the large square to the right of the sliders. 'New graphic' is written on this large square. The program will now calculate the light intensity at many different places on the road surface. The variation in light intensity is shown in a graph entitled variation in light intensity. Here you will see 3 curves.

- The pale-yellow curve is usually (but not always!) the highest. This curve shows the light intensity directly below the lampposts.
- The dark-yellow curve shows the variation in light intensity if you walk along at a quarter of the width of the road (in the key you see light intensity 0.25 width). We have set the width of the road at 8 meters, so the dark-yellow curve shows the light intensity when you walk along 2 meters away from the edge of the road.
- Lastly, the orange curve shows the light intensity in the center of the road (light intensity at 0.50 width), so in our case that's 4 meters away from the edge of the road.

The program assumes there is a row of lampposts on either side of the road and that the lampposts are positioned directly opposite one another.
Each graph shows the variation in light intensity between two adjacent lampposts. The values shown furthest left in the graph are the light intensities at the point where the first street lamp is located, and the values shown furthest right in the graph are the light intensities at the point where the next street lamp is located.

Task

26. Enter the values for *a*, *h* and *P* that we have already used before (30 meters for *a*, 5 meters for *h* and 1000 watts for *P*). Click on the large square and view the graphs. Compare the values for the minimum and maximum light intensity you calculated in questions 18/19 with the values in the graph.

Work out the difference between them and try to find an explanation for this.

Above the graphs you will see the values for the minimum and maximum light intensity required by law (10 and 320 watts/m^2). Once the three curves have been calculated it is easy to see whether the lowest light intensity remains above the legal minimum, and whether the highest light intensity remains below the legal maximum. For this purpose a dark-gray line has been drawn in the same graph to mark the minimum light intensity (10 watts/m^2), and a light-gray line marks the maximum light intensity (320 watts/m^2).

The *street_lighting_what_does_it_cost* program monitors whether the yellow and orange curves remain within the minimum and maximum values. If they do, the program shows *minimum approved* (on a green background) and *maximum approved* (also on a green background). However, if at any point the light intensity rises too high, *maximum too high* will appear on a red background, and if the light intensity is too low at any point *minimum too low* will appear on a red background. This means the solution in question is not acceptable.

You can see that with the values we have used the light intensities on the surface of the road are not too high or too low and that the program therefore accepts this solution.

The costs are shown in the yellow box next to the graph. All being well, these costs should correspond to the costs you calculated in question 24.

Later on in the worksheet all kinds of values have been entered, such as the price of one kWh (€ 0.15). You can change these values, but it is probably best to leave them as they are for the time being.

Have a go…

With this computer application finding the best solution is just a matter of having a go. Use the sliders to set values for *a*, *h* and *P*; click to calculate the curves and then check that both the minimum and the maximum are within the legal limits.

The aim is to find the lowest possible total cost per year per meter of road! The lower the overall cost, the cheaper your solution will be. You are only allowed to use solutions that lie within the legal minimum and maximum values.
The program continuously provides information about the cost. As soon as you assign a different value to one of the quantities *a*, *h* or *P* (for example by dragging one of the sliders) the total cost will be calculated again and shown in the yellow box.

… but think about it carefully!

In order to make the challenge a bit more exciting, we have made it into a competition. You will be playing against each other and the aim is to find the cheapest possible solution that complies with the minimum and maximum values. You are only allowed to click on the *new image* button forty times. That means you need to think very carefully and, particularly at the end, you must not just guess wildly at the values for *a*, *h* and *P*. Once you have experimented a bit to see the effects of increasing and decreasing *a*, *h* and *P*, try to predict what will happen when you enter a new value. Try to understand why the curves look the way they do – that's the only way to win! The *street_lighting_what_does_it_cost* program tells you how many attempts you have left. Your time is up when you see the message 'GAME OVER' ...

How cheaply can you light a road …?